Axially symmetric membranes with polar tethers

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Abstract

Axially symmetric equilibrium configurations of the conformally invariant Willmore energy are shown to satisfy an equation that is two orders lower in derivatives of the embedding functions than the equilibrium shape equation, not one as would be expected on the basis of axial symmetry. Modulo a translation along the axis, this equation involves a single free parameter c. If $c \neq 0$, a geometry with spherical topology will possess curvature singularities at its poles. The physical origin of the singularity is identified by examining the Noether charge associated with the translational invariance of the energy; it is consistent with an external axial force acting at the poles. A one-parameter family of exact solutions displaying a discocyte to stomatocyte transition is described.

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1 Introduction

The course-grained description of membranes and interfaces often involves purely geometrical degrees of freedom. The energy associated with tension is proportional to area. The positive definite invariant,

$$H = \frac{1}{2}\int dA\, K_{ab}K^{ab}\,, \tag{1}$$

quadratic in the surface extrinsic curvature K_{ab}, is a measure of the energy associated with bending [1] (the notation we adopt is defined in an appendix). This energy has the remarkable property that it is invariant with respect to conformal transformations of the ambient three-dimensional space. It plays a prominent role in the physics of soft matter, notably in the description of fluid membranes and liquid crystals (see, for example, [2, 3]). Physically realistic models will, of course, involve additions to H which break the conformal symmetry [4, 5, 6]: there may be tension and an osmotic pressure associated, respectively, with global constraints on the area and on the volume; there also may be an asymmetry reflected in an energy linear in curvature. There are, however, important lessons we can learn from the simple description

of the surface provided by Eq.(1). For it is still possible to constrain the geometry using local constraints which break the conformal invariance of Eq.(1) only at isolated points.

The Euler-Lagrange equation corresponding to Eq.(1) is given by (∇^2 is the surface Laplacian)

$$-\nabla^2 K + \frac{1}{2} K (K^2 - 2 K_{ab} K^{ab}) = 0\,. \tag{2}$$

The few known analytical solutions of Eq.(2) are easily identified by inspection: these are spheres with $K_{ab} = g_{ab} K/2$, as well as the minimal surfaces with $K = 0$. In these geometries the membrane is free of stress. There is also the well-known Clifford torus with ratio of wheel to tube radius of $\sqrt{2}$. In practice, one relies overwhelmingly on numerical analysis to map out the configuration space (some recent developments are described in [7, 8, 9]). On the other hand, in the early nineties Seifert, Julicher, Lipowsky and others pointed out features of solutions lying beyond the reach of numerical analysis that are direct consequences of the conformal symmetry of the energy [10]. After a brief flurry of activity the subject waned. The goal of this paper will be to underscore the point that the conformal symmetry of Eq.(1) merits reexamination.

We will first show that, modulo a translation parallel to its axis, all axially symmetric equilibrium shapes satisfy the following equation

$$\frac{R^2}{2}\left(C_\perp^2 - C_\parallel^2\right) - c\,\mathbf{n}\cdot\mathbf{X} = 0\,. \tag{3}$$

Here $C_\perp$ and $C_\parallel$ are the principal curvatures along the meridian and along the parallel, R is the polar radius, and $\mathbf{n}\cdot\mathbf{X}$ is the support function. This equation is two orders lower in derivatives of the embedding functions $\mathbf{X}$ than Eq.(2), not one derivative, as would have been expected on the basis of axial symmetry. It involves a single free parameter c.

The conformal invariance of the energy may be exploited to construct a one-parameter family of solutions of Eq.(3). This symmetry implies that an equilibrium geometry maps to another equilibrium geometry under conformal transformation. A catenoid is a minimum surface and thus an equilibrium. By inverting the catenoid in points along its axis a sequence of deflated compact axially symmetric equilibrium geometries displaying a discocyte to stomatocyte transition is generated.

It is also possible to identify important properties of solutions of Eq.(3) without having to solve it explicitly. The only solution with spherical topology that is regular at its poles is the round sphere with $c = 0$. If $c \neq 0$, however, all solutions with this topology exhibit curvature singularities at their poles. In general, this parameter determines the strength of the singularity. The physical interpretation of c is as an external force pulling the poles of the membrane together. This identification is based on the existence of a conserved stress tensor on the membrane associated with translational invariance. The forces are the Noether charges associated with this symmetry [11]. The potential relevance of these solutions to modeling tethered membranes will be discussed.

2 Axially symmetric equilibrium shapes

We begin by deriving Eq.(3). Our approach will be to adapt the variational principle to a parametrization that exploits both the conformal symmetry of the energy as well as the axial symmetry of the configuration. Technically, this is simple. However, it is also all too easy to overlook constraints or to lose sight of the dependence on the geometry implicit in the parametrization. It is therefore useful to go through the details carefully. The equation itself will be independent of the parametrization.

An axially symmetric surface is described in terms of cylindrical polar coordinates (ρ, z, ϕ) on R^3. The axis of symmetry coincides with the z-axis. There are two parametrizations of the surface tailored to this symmetry that we will find useful. The more straightforward of the two involves arc-length l along the meridian and the angle ϕ along the parallels. The surface is then described by two functions $\rho = R(l)$, and $z = Z(l)$; these functional relationships are not independent; they satisfy the constraint $R'^2 + Z'^2 = 1$ implied by the parametrization, where the prime denotes a derivative with respect to arc-length. The line element on the surface is now given by

$$ds^2 = dl^2 + R(l)^2 d\phi^2 \,. \tag{4}$$

An isothermal parametrization is obtained by replacing arc-length l by a parameter u defined in such a way that the line element assumes the conformally flat form,

$$ds^2 = e^{2\sigma(u)}(du^2 + d\phi^2) \,. \tag{5}$$

Comparison of the two expressions (4) and (5) identifies $R = e^\sigma$, and connects the parameter u to arclength through the relation $\partial_u l = R$.

To characterize the extrinsic geometry, introduce the variable Ω representing the angle which the outward normal to the surface makes with the (positive) axis of symmetry; the tangent along the meridian $\mathbf{l}$ is then given by $(R', Z', 0) = (\cos\Omega, -\sin\Omega, 0)$. At any point on the surface, $\mathbf{l}$ together with the tangent along the parallel $\mathbf{t}$ define the principal axes of the surface. The corresponding curvatures are given respectively by

$$C_\perp = \Omega' = \frac{\partial_u \Omega}{R} \,, \qquad C_\parallel = \frac{\sin\Omega}{R} \,. \tag{6}$$

As noted in [12], in terms of isothermal coordinates, the bending energy then assumes the deceptively simple form

$$H[\Omega] = 2\pi \int du \, \mathcal{H}_0 \,, \tag{7}$$

where

$$\mathcal{H}_0 = \frac{1}{2} \left\{ (\partial_u \Omega)^2 + \sin^2\Omega \right\} \,. \tag{8}$$

In particular, H appears to depend only on Ω. Importantly, σ does not appear in H, at least not explicitly. This is a direct consequence of the global conformal invariance of the energy [1].

To determine the Euler-Lagrange equation for Ω, it is necessary to impose constraints on Ω to reflect the fact that Ω parametrizes the normal vector. The variational principle must therefore be consistent with the following local constraints,

$$\cos\Omega = R^{-1}\partial_u R\,, \quad \sin\Omega = -R^{-1}\partial_u Z\,. \tag{9}$$

We thus introduce two local Lagrange multipliers λ_R and λ_Z (as we see, one is generally not enough) and replace $\mathcal{H}_0$ by $\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H}$ where

$$\Delta\mathcal{H} = \lambda_R(\cos\Omega - R^{-1}\partial_u R) + \lambda_Z(\sin\Omega + R^{-1}\partial_u Z)\,. \tag{10}$$

The Euler-Lagrange equations for R and Z give

$$\partial_u\lambda_R + \sin\Omega\,\lambda_Z = 0\,, \tag{11}$$

and

$$\partial_u\lambda_Z - \cos\Omega\,\lambda_Z = 0 \tag{12}$$

respectively. They do not involve $\mathcal{H}_0$. As such, they are easy to solve. The solution is

$$\lambda_Z = c\,R\,, \quad \lambda_R = c\,Z + d\,; \tag{13}$$

it involves two constants of integration, c and d. Note that if the latter constraint is overlooked, we miss the constant c. This oversight is thus equivalent to setting $c = 0$. The role of the constant d is trivial; it is clear that we can always set $d = 0$ when $c \neq 0$ by translating the geometry along the axis of symmetry.

The Euler-Lagrange equation for Ω is now given by

$$-\partial_u^2\,\Omega + \sin\Omega\cos\Omega - \lambda_R\,\sin\Omega + \lambda_Z\,\cos\Omega = 0\,. \tag{14}$$

Ω couples to the intrinsic geometry through the two multipliers λ_R and λ_Z given by Eq.(13). Using the constraints we can cast λ_R and λ_Z non-locally in terms of Ω. This has the appearance of a non-local generalization of the sine-Gordon equation. However, as we will show this analogy is not very useful. One can do much better.

If $c = 0$, then λ_Z vanishes and λ_R is constant. Ω then uncouples from the intrinsic geometry and one obtains a (local) double sine-Gordon equation. Such an equation was obtained recently [13]. The non-local terms are removed. We will show, however, that this limits the space of solutions very severely: only spheres and catenoids remain.

Remarkably, it is possible to integrate Eq.(14) to produce a ‘quadrature’:

$$\frac{1}{2}(\partial_u\Omega)^2 - \frac{1}{2}\,\sin^2\Omega - \lambda_R\cos\Omega - \lambda_Z\sin\Omega = E\,. \tag{15}$$

The technical reason why this is possible is because the two derivative terms involving the multipliers cancel:

$$\cos\Omega\,\partial_u\lambda_R + \sin\Omega\,\partial_u\lambda_Z = 0\,. \tag{16}$$

There is a deeper geometrical reason which will be developed below. The result is that a quadrature is obtained even when $c \neq 0$ so that neither λ_R nor λ_Z is constant. Of course, in this case, unlike the sine-Gordon equation, it will not generally be possible to integrate the quadrature.

There remains to address a subtlety associated with reparametrization invariance that has been carefully sidestepped so far: specifically, in the variational principle, it is necessary to take into account the fact that the isothermal parametrization is not some arbitrary parametrization of the surface: it depends implicitly on the surface geometry. Physical solutions will need to be consistent with the constraint implied by this dependence. As a consequence, the constant of integration E entering Eq.(15) must vanish.

To see this, note that if the interval of integration $u_f - u_i$ is fixed, a global constraint

$$\int \frac{dl}{R} = u_f - u_i \tag{17}$$

is placed on the geometry — a constraint which is clearly unphysical. While it does respect scale invariance, this constraint spoils the conformal invariance of the problem. Thus if the position along the meridian is parametrized by the isothermal variable u, u itself must be allowed to vary freely at the endpoints within the variational principle. Now, modulo Eqs.(9), (13) and (14), it is easily checked that $\delta H = 2\pi(E(u_f)\,\delta u_f - E(u_i)\,\delta u_i)$, where

$$E(u) = \frac{\partial \mathcal{H}}{\partial\, \partial_u \Omega}\,\partial_u \Omega + \frac{\partial \mathcal{H}}{\partial\, \partial_u R}\,\partial_u R + \frac{\partial \mathcal{H}}{\partial\, \partial_u Z}\,\partial_u Z - \mathcal{H}\,. \tag{18}$$

Therefore, in a stationary configuration, $E(u)$ must vanish at the endpoints. And because $\mathcal{H}$ does not depend explicitly on u, E must be constant. As a result, it is zero everywhere. This is, of course, the constant appearing in the quadrature (15). In the appendix, we will show that E can be identified as the global Lagrange multiplier associated with the constraint (17). When E vanishes, the unphysical constraint is relaxed. This, of course, is not new: a completely analogous problem is encountered in the parametrization of an axially symmetric geometry by arc-length [14]; it is a subtlety that is perhaps more familiar in the modeling of reparametrization invariant relativistic systems. In this context it is often desirable to introduce a parameter depending explicitly on the trajectory, such as proper time, within the variational formulation of the dynamics (see, for example, [16]). The issue is discussed in the Hamiltonian setting in [17].

The beauty of Eq.(15) is not its sine-Gordon form, but its remarkably simple gauge invariant geometrical structure which is manifest when it is expressed in terms of the principal curvatures and the support function $\mathbf{n} \cdot \mathbf{X}$ to give Eq.(3).

In the next section, we will look at equilibrium from the point of view of the stresses that exist within the membrane; this approach will permit us to interpret the constant c in terms of external forces. Such an interpretation is not obvious within the simple economical framework we have been using up to now. It will also permit us to establish the consistency of Eq.(3) with the corresponding first integral of the shape equation (2). What will become clear is that no single approach is appropriate everywhere.

3 Stress and Noether charges

The statement of equilibrium can be cast in terms of a conserved stress tensor [11]. This framework does, unfortunately, involve the introduction of a little extra formalism.

Using the notation defined in the appendix, the stress in the membrane is given by [11]

$$\mathbf{f}^a = K(K^{ab} - \frac{1}{2}g^{ab}K)\,\mathbf{e}_b - \partial^a K\,\mathbf{n}\,. \tag{19}$$

A simple derivation is provided in [18]) using auxiliary variables. It has also been subject of a recent detailed treatment [19]. On the free surface of the membrane, $\mathbf{f}^a$ is conserved in equilibrium so that

$$\nabla_a\,\mathbf{f}^a = 0\,, \tag{20}$$

where ∇_a is the surface covariant derivative compatible with g_{ab}. This description is manifestly invariant with respect to reparametrizations of the surface. One can easily check that Eq.(20) reproduces the shape equation (2). From a field theoretical point of view, Eq.(20) is a direct consequence of the translational invariance of the energy. $\mathbf{f}^a$ is a Noether current. We will now show that the constant c appearing in Eq.(3) is none other than the corresponding Noether charge.

In reference [11], the global statement of conservation which follows from Eq.(20) was discussed. The focus, however, was on the behavior of an isolated regular topologically spherical membrane subject to global internal constraints; thus the boundary conditions that were admitted were not the most general. They are certainly not the boundary conditions that are appropriate here. For this reason, it will be useful to retrace the steps involved in integrating the conservation law to accommodate curvature singularities or non-trivial boundary conditions, and to provide an interpretation.

Thus, consider any closed contour Γ on the free surface of a membrane. Using Stoke's theorem, the conservation law Eq.(20) implies that the closed line integral

$$\oint ds\; l^a\,\mathbf{f}_a \tag{21}$$

is a constant vector $\mathbf{F}$ along contours that are homotopically equivalent to Γ on this surface. Here l^a is the normal to Γ tangent to the surface, and ds is the element of arclength. If the geometry is regular and the contour can be contracted to a point this vector vanishes. There are, however, various possible obstructions: the membrane topology may be non-trivial, as it is for a torus. There may also be a source of stress within the region spanned by the contour. In the latter case the geometry may be regular if the source is distributed over a finite area. A idealized point source of stress, however, will be reflected in a curvature singularity which is picked up by the line integral. $\mathbf{F}$ has a clear interpretation as the net force acting on the region [20].

In an axially symmetric geometry, this force can only be parallel to the axis. We thus have ($\mathbf{k}$ is a unit vector pointing along the z axis)

$$\oint ds\; l^a\,\mathbf{f}_a\cdot\mathbf{k} = 2\pi c\,, \tag{22}$$

where c is a constant.

In an axially symmetric geometry, it is particularly simple to evaluate the integrand appearing in Eq.(22). Without loss of generality, let the contour be a closed circle of fixed Z. Then l^a is the tangent to the meridian ($\mathbf{l} = l^a \mathbf{e}_a$) and ds is the element of arclength along the parallel $ds = Rd\phi$. Now

$$l_a \mathbf{f}^a = \frac{1}{2}(C_\perp^2 - C_\parallel^2)\,\mathbf{l} - (C_\perp + C_\parallel)'\,\mathbf{n}\,, \tag{23}$$

so that

$$l_a \mathbf{f}^a \cdot \mathbf{k} = \frac{1}{2}\Big((\partial_u \Omega)^2 - \sin^2\Omega\Big)\,\frac{\sin\Omega}{R^2} + \,\Big(\partial_u^2 \Omega - \sin\Omega\cos\Omega\Big)\,\frac{\cos\Omega}{R^2}\,. \tag{24}$$

The integrand appearing on the lhs of (22) does not depend on ϕ.

To establish consistency with the approach adopted in the previous section, note that there is a linear combination of the Euler-Lagrange equation (14) and its 'first integral' (15) which permits the elimination of λ_R:

$$\frac{1}{2}\sin\Omega\,\Big((\partial_u \Omega)^2 - \,\sin^2\Omega\Big) + \cos\Omega\,\Big(\partial_u^2\Omega - \frac{1}{2}\sin 2\Omega\Big) - \lambda_Z = 0\,. \tag{25}$$

It should now be clear that equation (25) coincides with the statement of conservation for the underlying stress tensor. It is the first integral of the shape equation. Consistency requires the identification of the two constants of integration. The framework of a conservation law provides a remarkably direct interpretation of the constant c appearing in Eq.(3).

We note one important point: had one insisted on holding on to a non-vanishing value of E in equation (25) it would not have been consistent with (22). Consistency requires both $E = 0$ and the identification of the constant of integration c as anticipated. In the appendix, we also show that the unphysical constraint on the range of the isothermal parameter u manifests itself in this framework through an additional unphysical tangential stress.

It should be pointed out that we have not found a simple derivation of Eq.(3) that follows directly from Eq.(2). The first integral of the latter, described above, is a weaker statement than Eq.(3) for it involves one extra derivative.

Of course, the existence of a first integral of the shape equation for axially symmetric geometries is well known using alternative approaches [14, 15]. The appearance of a constant of integration c was also noted. As Zheng and Liu, among others, have emphasized the physics does not depend on the parametrization [15]. However, the physical role of this constant does not appear to have been appreciated. Posing the problem in terms of a conservation law is not just a formal exercise. What was not anticipated was that it would be possible to do much better than Eq.(25). For, as we have seen, we possess the stronger result, Eq.(3) involving not one but two integrations of Eq.(2).

Before describing new solutions of Eq.(3), it is a useful exercise to identify various features of the stress tensor in the familiar regular axially symmetric geometries: In the case of the catenoid, a minimal surface with $C_\perp + C_\parallel = 0$, it is obvious from Eq.(3) that $c = 0$. On the other hand, the constant c has a non-vanishing value on the Clifford torus. Consider a circular torus characterized by two radii, a and b,

$$R = a + b\sin\Omega\,. \tag{26}$$

Note that we now have $l = b\,\Omega$, so that $\partial_u \Omega = R/b$. It is now simple to check that R is a solution of Eq.(3) if and only if $a = \sqrt{2}b$. The value of c is fixed, $c = 1/b$ (and, of course $E = 0$). In particular, c does not vanish. The origin of the constant c for the Clifford torus lies clearly in its topology. Observe that whereas the stress tensor itself, given by Eq.(19) vanishes everywhere on spheres and minimal surfaces, it does not on the Clifford torus.

4 New Solutions

The conservation law provides the appropriate framework for characterizing equilibrium geometries. If the parameter c appearing in this equation does not vanish, it is simple to see that any topologically spherical solution of Eq.(3) must possess a curvature singularity at its poles. The strength of this singularity is proportional to c. Furthermore, the round sphere is the only non-singular solution of Eq.(3) with this topology. For if the geometry at either pole is regular and lies on the free membrane, then the contour determining c can be contracted onto that pole. The constant must therefore vanish: $c = 0$. The boundary condition at the pole, $\partial_u \Omega = 0 = \sin\Omega$, then implies that $d = 0$ in Eq.(15). As a consequence $C_\parallel = C_\perp$ on the free surface. It is thus necessarily a part of a sphere.

The conformal invariance of the energy permits us to identify a one-parameter family of solutions of Eq.(15). We note that, under inversion in the origin,

$$\mathbf{X} \to \mathbf{X}/|\mathbf{X}|^2 \,, \tag{27}$$

a minimal surface satisfying $K = 0$, is mapped to a surface satisfying

$$K = 4\frac{\mathbf{n}\cdot\mathbf{X}}{|\mathbf{X}|^2} \,. \tag{28}$$

This surface will generally possess a curvature singularity at the origin if this point lies on the surface. It will also describe an equilibrium as a consequence of conformal invariance. Thus, if it is also axially symmetric, it must satisfy Eq.(3), except perhaps at the origin. However, this equation can also be cast as

$$\frac{R^2}{2}\left(C_\perp - C_\parallel\right) K - c\,\mathbf{n}\cdot\mathbf{X} = 0 \,. \tag{29}$$

Together Eqs. (28) and (3) imply

$$\frac{2R^2}{|\mathbf{X}|^2}\left(C_\perp - C_\parallel\right) = c \,. \tag{30}$$

The only axially symmetric minimal surfaces are the catenoids, given parametrically by [1]

$$R(\Omega) = R_0^{-1}\,\csc\Omega \,, \quad Z(\Omega) = R_0^{-1}(\ln\tan\Omega/2 + \xi_0) \,, \tag{31}$$

[1]Scaling the catenoid with an inverse length will give an inverted geometry with the 'correct' dimensions.

or, equivalently, as the level set

$$R - R_0^{-1} \cosh(R_0 Z + \xi_0) = 0 \,. \tag{32}$$

It is characterized by a scale $1/R_0$ and an offset ξ_0/R_0 along the axis. All catenoids are related by scaling and translation. Inverted in the origin, catenoids map to the surfaces described by the following transcendental equation

$$\frac{R}{R^2+Z^2} - \frac{1}{R_0} \cosh\left(\frac{R_0 Z}{(R^2+Z^2)} + \xi_0\right) = 0 \,. \tag{33}$$

It is simply to check that these surfaces are indeed solutions to Eq.(3). Remarkably, each value of the displacement of the catenoid ξ_0 with respect to the point of inversion describes a distinct equilibrium.

All of these geometries are compact. The two poles touch because all distant parts of the original catenoid are mapped to the origin, a peculiarity of our exact solution. However, a tangent plane does exist at the origin and one can check that the Euler characteristic is consistent with a sphere. Their topology, despite appearances, is spherical.

If $\xi_0 = 0$, the surface is a symmetric biconcave geometry: a discocyte. By translating ξ_0 along the axis of symmetry, a physically interesting one-parameter family of solutions is generated; the up-down symmetry gets broken and there is a transition from a discocyte to a stomatocyte. Inversion has connected geometries describing very distinct physical scenarios. The geometric profiles for three different values of this offset are illustrated in Fig.(1). The profiles have been rescaled so that they possess the same surface area.

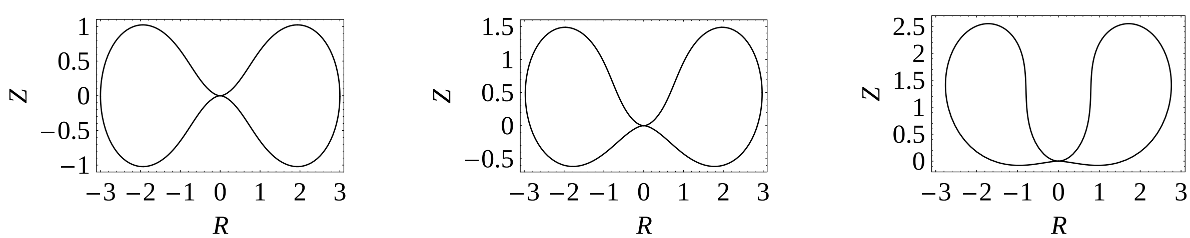

Figure 1: Geometric profiles for $\xi_0 = 0, 1, 2$.

There are superficial similarities with the well-known discocyte to stomatocyte transition induced by a change in the bilayer asymmetry (see, for example, [21, 5, 6]). However, whereas conformal symmetry is broken globally in that model, here it is broken only locally (at the poles). The appropriate parameter describing this transition is the interpolar tension. Its properties will be discussed in detail in a forthcoming paper. We will thus limit ourselves to a brief description of the symmetric discocyte geometry.

Whereas a minimal surface (with $K = 0$) is unstressed, as Eq.(19) demonstrates, the inverted state in under stress. In the symmetric biconcave geometry, it can be shown that the

two principal curvatures diverge as

$$R_0 C_{\|}\ , R_0 C_{\perp} \approx -2\ln\left(R/R_0\right) \tag{34}$$

The poles are umbilical points of the geometry, albeit singular ones. It is clear from Eq.(30) that $C_{\perp} - C_{\|}$ will generally be finite.

The curvature singularity at the poles is a manifestation of local force tethering the poles together. It is unrelated to the fact that they touch. The magnitude of this force is given by the Noether charge. A straightforward calculation gives

$$c = -4/R_0 \,.$$

The sign indicates that the force is directed towards the interior as we anticipated.

The exact solutions we have written down have their poles joined. More generally, solutions of Eq.(3) exist with poles a fixed distance apart. Like our exact solutions, they will be singular at these poles.

The tether fixing this distance permits a constraint to be placed on the isoperimetric ratio: the volume is necessarily deflated below the maximum spherical value. Remarkably, it is possible to constrain this ratio without mutilating the conformal invariance of the energy. This is because the source breaking this symmetry acts only at isolated points.

It may appear that the solutions we have described with tethered poles are unphysical. However, even apart from the tethers that are pulled in the micromanipulation of membranes (see, for example, [22, 23]), it should be emphasized that tethers are ubiquitous features in biological membranes. For example, it is known that the Golgi complex is not an isolated equilibrium structure. It is stabilized against breakup by a network of microtubules that are dismantled and reassembled during mitosis. The tethered membrane we have described is one of those rare tractable toy models which is relevant in this context.

5 Conclusions

A new equation describing the equilibrium of an axially symmetric fluid membrane has been presented. While our derivation exploits a specific adapted coordinate system, it is possible to cast the equation in a form involving only geometrically significant quantities: the principal curvatures, the support function as well as the radial distance function. The equation has the remarkable property that it involves two less derivatives of the embedding functions than does the corresponding shape equation. This behavior is unexpected. Even in this simple scenario there is more to be understood.

We have identified new solutions to the axially symmetric shape equation under the influence of external forces with physical relevance. This opens a non-perturbative analytical window on processes that are important in biophysics and soft matter. Work in this direction is in progress.

Even if one is not interested in the construction of axially symmetric equilibrium shapes, the problem we have addressed displays several features of interest from a geometrical point of

view. In particular, the implementation of the variational principle presents genuine subtleties with a combination of physical constraints that need to be enforced and unphysical constraints that need to be relaxed. How to negotiate these points without running into errors is of broader relevance than the specific problem being addressed.

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APPENDIX

Notation

A surface is described by three functions $\mathbf{X} = (X^1, X^2, X^3)$ of two variables ξ^1, ξ^2. The two coordinate tangent vectors to the surface are given by $\mathbf{e}_a = \partial_a \mathbf{X}$, $a = 1, 2$ ($\partial_a = \partial/\partial\xi^a$). Let $\mathbf{n}$ be the unit normal. The metric tensor induced on the surface and the extrinsic curvature are then given in terms of these vectors by $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$ and $K_{ab} = \mathbf{e}_a \cdot \partial_b \mathbf{n}$ [24, 25, 26]. Indices are raised with the inverse metric. $dA = \sqrt{\det g_{ab}} d^2\xi$ is the area measure induced on the surface by $\mathbf{X}$. K denotes (twice) the mean curvature: $K = g^{ab} K_{ab}$

Recovering the shape equation

It is simple to confirm that a first derivative of equation (25) gives

$$(-\partial_u + \cos\Omega)\left(\partial_u^2\Omega - \sin\Omega\cos\Omega\right) - \frac{1}{2}\left((\partial_u\Omega)^2 - \sin^2\Omega\right)(\partial_u\Omega - \sin\Omega) = 0\,. \qquad (35)$$

A short calculation (substituting Eq.(6) into Eq.(2)) confirms that this equation is the axisymmetric shape equation (2) in isothermal parametrization. It is also evident that the first term appearing in Eq.(35) coincides with the 'kinetic' term in (2) whereas the second term coincides with the cubic potential. This establishes explicitly the consistency with the reparametrization invariant shape equation.

Unphysical tangential stress associated with constraint on u

We will show that a non-vanishing value for the constant E appearing in the quadrature, associated with an unphysical constraint on the range of u, has its unphysical counterpart in

the stress tensor. If this constraint is introduced with a Lagrange multiplier $\tilde{E}$, it will add a tangential contribution to the surface stress, given by

$$\mathbf{f}^a_{\text{constant}\,u} = -\tilde{E}\,\frac{\mathbf{e}^a}{R^2}\,. \tag{36}$$

This is a surface tension with an unusual inverse R^2 dependence. If this stress is added to $\mathbf{f}^a$ consistency with Eq.(25) requires $E = \tilde{E}$. The constraint on u introduces tension. It is not however the usual surface tension. The energy associated with a constant surface tension μ is proportional to area and the corresponding stress is $\mathbf{f}^a_{\text{tension}} = -\mu\,\mathbf{e}^a$.

The easiest way to derive Eq.(36) is to recast the integral appearing in Eq.(17) in the equivalent form,

$$\frac{1}{2\pi}\int \frac{dA}{R^2}\,. \tag{37}$$

Adding such a term breaks the translational invariance of the energy. Consequently, the associated stress will not be conserved. The energy remains, however, invariant under translations parallel to the axis. Thus, the projection of the stress along the axis $\mathbf{f}^a\cdot\mathbf{k}$ is still conserved. Using the auxiliary variables introduced in [18], it is straightforward to show that the corresponding addition to the stress is the one given by Eq.(36). For the record, the conservation law (20) is replaced by

$$\nabla_a\,\mathbf{f}^a = \frac{4E}{R^4}\,\left(\mathbf{X} - (\mathbf{X}\cdot\mathbf{k})\,\mathbf{k}\right)\,. \tag{38}$$

This description of stress conservation is invariant under reparametrization; one need not worry whether the coordinates we have chosen are well-behaved under variation.

For the same reason as a constraint on u induces an unphysical stress one also needs to be careful when parametrizing the surface geometry by arc-length. Instead of constraining the integral appearing in (37), one has a constraint on $\int dA/R$. The corresponding stress $\mathbf{f}^a_{\text{constant}\,l}$ is then proportional to $\mathbf{e}^a/R$. This constraint is relaxed in a manner entirely analogous to that employed for the isothermal parameter u.

Finally, we comment that non-trivial equilibrium single attached particle configurations do not exist without tension. For, as we have shown, if $E = 0$, then the only solution consistent with regular geometry at the south pole is part of a sphere.

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